§ 2.1 Superconformal Multiplets and Supersymmetric Deformations

$\{Q_i, Q_j\} \sim 0$, $\quad i,j = 1, \dots, N_Q$

Superconformal multiplet: $Q^\ell \mathcal{V}$,

where $0 \le \ell \le l_{max}$, $\quad l_{max} \le N_Q$

Long multiplets

no null-states

$\mathcal{V}$ is irreducible under Lorentz- and R-sym.

at level ℓ: $\quad \left(\Lambda^\ell R_Q\right) \otimes \mathcal{V} \qquad (*)$

where R_Q is the Lorentz- and R-sym rep of Q

→ unique top component: $l_{max} = \dim R_Q = N_Q$

→ Lorentz- and R-sym. singlet

dimension: $\Delta_\mathcal{V} + \frac{1}{2} N_Q$

- Racah-Speiser algorithm:
  - select highest-weight state $\mathcal{V}_{h.w.} \in \mathcal{V}$ with respect to L- and R-sym.
  - at each level ℓ, consider all sequences of ℓ supercharges acting on $\mathcal{V}_{h.w.}$

$$Q_{i_1} Q_{i_2} \cdots Q_{i_\ell} \mathcal{V}_{h.w.}$$

add L- and R-sym. weights of Q_i to

those of $\mathcal{V}_{h.w.}$

→ set of RS trial weights $\mathcal{W}_{RS}^{(\ell)}$ at level ℓ

→ one-to-one correspondence between irreducible reps. of (*) and RS trial weights $\mathcal{W}_{RS}^{(\ell)}$

- when $\mathcal{V}$ is too small, the bijection can fail

Example:

consider long multiplets in 3d $\mathcal{N}=1$ SCFTs

→ R-sym. is trivial, L-sym. is $su(2)$

supercharges Q_α $(\alpha = \pm)$ transform as L-doublet

notation: $R_Q = [1]$

If L-rep. of $\mathcal{V}$ is $[n]$ → n-index symmetric spinor $\mathcal{V}_{(\alpha_1, \dots, \alpha_n)}$ with $\alpha_{i=1,\dots,n} = \pm$

$$\ell = 0 : \quad \mathcal{V}_{h.w.} = \mathcal{V}_{++\cdots+}, \quad \mathcal{W}_{RS}^{(0)} = \{[n]\}$$

$$\ell = 1 : \quad Q_+\mathcal{V}_{h.w.}, Q_-\mathcal{V}_{h.w.}, \quad \mathcal{W}_{RS}^{(1)} = \{[n+1], [n-1]\}$$

$$\ell = 2 : \quad Q_+ Q_- \mathcal{V}_{h.w.}, \quad \mathcal{W}_{RS}^{(2)} = \{[n]\}$$

For $n \geq 1$, we have a match between $\mathcal{W}_{RS}^{(\ell)}$ and irreps of conformal primaries

For $n=0$ only $[1]$ rep. appears at level 1

$[-1]$ is removed by RS algorithm

RS trial states do not coincide with <u>true</u> highest weight states of corr. rep. :

true highest weight at $l=1$ of $[n-1]$ is

$$Q_- V_{++\cdots+} - Q_+ V_{-+ +\cdots +} \neq Q_- V_{++\cdots+}$$

(RS algorithm bypasses full Clebsch-Gordan problem)

$$Q_- V_{+++\cdots+} = \frac{1}{2}\Big[\text{highest weight}\big([n-1]\big) + \text{h.w.}\big([n+1]\big)\Big]$$

$\rightarrow$ Q_- does <u>not</u> take us from $[n]$ to $[n-1]$

<u>transition $l \rightarrow l+1$:</u>

suppose $\mathcal{O}$ is conformal primary at level l

$Q\,\mathcal{O} \longrightarrow \mathcal{O}'$ at level $l+1$

$$\mathcal{O}' \subset R_Q \otimes \mathcal{O}$$

However: $\mathcal{O}'$ might not be in the image of Q!

1) due to Fermi-statistics or if multiplet has null-states

2) $\mathcal{O}'$ occurs at level $l+1$, <u>but</u> transition does not occur!

example:

consider long multiplet in 3d $\mathcal{N}=4$ SCFT

$\rightarrow$ $Q_\alpha^{i,i'}$ transform in trifundamental $[1]_{1/2}^{(1,1)}$

of $SU(2)_R \times SU(2)'_R$ and $SU(2)$ L-sym.

take $\nu \in [0]^{(0;0)}$ to be a singlet

true highest weights:

$$\ell=2: \quad S = \left(Q_+^{++} Q_-^{--} + Q_+^{--} Q_-^{++} - Q_+^{+-} Q_+^{-+} - Q_+^{-+} Q_-^{+-}\right)\nu \in [0]^{(0;0)}$$

$$\ell=3: \quad \mathcal{O} = Q_+^{++} S \in [1]^{(1;1)}$$

$$\ell=4: \quad \mathcal{O}' = Q_+^{++} Q_+^{+-} Q_+^{-+} Q_-^{++} \nu \in [2]^{(2;2)}$$

Note that $Q: \mathcal{O} \to \mathcal{O}'$ does not occur because

$$Q_+^{++}\mathcal{O} = (Q_+^{++})^2 S = 0 \text{ by Fermi statistics}$$

although $[2]^{(2;2)} \in R_Q \otimes \mathcal{O} = [1]^{(1;1)} \otimes [1]^{(1;1)}$

<u>short multiplets:</u>

possess null states $\rightarrow$ must be removed from representation

We have the following possibilities for top components:

1) "Manifest top components":
conformal primary $\mathcal{O}$ with $Q\mathcal{O}$ = descendant
i.e. no conformal primary at level $\ell+1$ in tensor product $R_Q \otimes \mathcal{O}$

examples: - all primaries at level ℓ_{max}
- universal mass deformation in 3d discussed in last lecture

2) "Accidental top components":

there are conformal primaries at level $\ell+1$ but they not in the image of $Q|_{\text{level } \ell}$

We will focus on class 1)

Consider a generic short multiplet in 3d $\mathcal{N}=2$

Q_α, $\bar{Q}_\alpha$ carry $U(1)_R$ charges -1 and $+1$

$$\longrightarrow [1]^{(-1)}_{1/2} \oplus [1]^{(1)}_{1/2}$$

$$\begin{tikzcd}
 & \boxed{[j]^{(0)}_{\frac{1}{2}j+1}} \arrow[dl, "Q"'] \arrow[dr, "\bar{Q}"] & \\
\boxed{[j+1]^{(-1)}_{\frac{1}{2}(j+3)}} \arrow[dr, "\bar{Q}"'] & & \boxed{[j+1]^{(+1)}_{\frac{1}{2}(j+3)}} \arrow[dl, "Q"] \\
 & \boxed{[j+2]^{(0)}_{\frac{1}{2}j+2}} &
\end{tikzcd}$$

$A_1\bar{A}_1$-type multiplet

unique operator at $l_{max} = 2$

case $j=2 \longrightarrow$ superconformal stress tensor example with two top components $A_2\bar{A}_2[0]_1^{(0)}$:

$$
\begin{tikzcd}
 & \boxed{[0]_1^{(0)}} \arrow[dl, "Q"'] \arrow[dr, "\bar{Q}"] & \\
\boxed{[1]^{(-1)}} \arrow[dr, "\bar{Q}"'] & & \boxed{[1]^{(+1)}} \arrow[dl, "Q"] \\
 & \boxed{[0]_2^{(0)} \oplus [2]_2^{(0)}} &
\end{tikzcd}
$$

$[2]_2^{(0)}$: conserved flavor current

$[0]_2^{(0)}$: Lorentz-invariant flavor mass deformation